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ON THE MOTION OF HYPERION.

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1. If we assume that the planes of the orbits of Titan and Hyperion coincide, the differential equations for the motion of the latter may be written

$$r^2 \frac{dw}{dt} = C + m'k^2 \int Srdt, \tag{1}$$

$$\frac{d^2r}{dt^2} - r\left(\frac{dw}{dt}\right)^2 + \frac{k^2}{r^2} = m'k^2R; \tag{2}$$

in which r and w are the radius vector and longitude in orbit of Hyperion, k the constant of the Saturnian system as derived from the orbit of Titan, m' the mass of Titan, $m'k^2R$ and $m'k^2S$ the components, in the plane of the orbit, of the disturbing force in the direction of and perpendicular to the radius vector, and C is a constant of integration.

We shall assume that r, w, R, and S can be developed into series of sines and cosines of multiples of $\theta = l' - l - \lambda$, having coefficients which are periodic functions of $\eta = 4l - 3l'$, in which l and l' are the mean longitudes in orbit of Hyperion and Titan, and λ is a constant to be derived from observation. η is evidently the longitude of the mean conjunction point of the two satellites, and $l' - l$ their mean elongation.

If we put

$$\mathrm{n} = \frac{k}{a^{\frac{3}{2}}},$$

and

$$k\sqrt{a(1-\nu)} = C + m'k^2 \int (Sr)_0 dt;$$

in which a, the mean distance of Hyperion from Saturn, is a constant, and $(Sr)_0$ is that portion of Sr which is independent of θ. ν is thus a function of η only.

We may also put

$$d\eta = (4n - 3n')\,dt,$$

and

$$C = k\sqrt{a(1-\nu_0)};$$

whence, if a', the semimajor axis of Titan's orbit, be taken as the unit of distance, we may write

$$\sqrt{(1-\nu)} = \sqrt{(1-\nu_0)} + \frac{m'}{\sqrt{(1+m')}}\frac{n'}{4n-3n'}\frac{1}{\sqrt{a}}\int (Sr)_0 d\eta,$$

which may be solved by a simple integration.

We may now write

$$r^2\frac{dw}{dt} = k\sqrt{a(1-\nu)} + \frac{m'}{1+m'}\,\frac{n'^2}{n'-n}\,U, \tag{3}$$

in which U is that portion of $\int Srd\theta$ which remains after omitting those terms which are independent of θ. As is evident,

$$d\theta = (n'-n)\,dt.$$

Substituting in (2) the value of dw/dt derived from (3), the former equation becomes

$$\frac{d^2r}{dt^2} - \frac{k^2a(1-\nu)}{r^3} + \frac{k^2}{r^2} = m'\,\frac{k^2}{a^2}\,P, \tag{4}$$

where, neglecting terms in m'^2,

$$P = a^2R + \frac{2\sqrt{(1-\nu)}}{s^3\sqrt{a}}\,\frac{n}{n'-n}\,U, \tag{5}$$

in which

$$s = \frac{r}{a}.$$

If we multiply by a^2/k^2, (4) becomes

$$\mu\frac{d^2s}{d\theta^2} + \frac{s-1+\nu}{s^3} = m'P, \tag{6}$$

in which

$$\mu = \frac{(n'-n)^2}{n^2}.$$

2. As the equations of motion of an intermediate orbit, we shall assume

$$r_{,} = a_{,}s_{,},$$

$$3(n'-n) = \frac{k^2}{a_{,}^3},$$

$$s_{,}^2\,\frac{dw_{,}}{3d\theta} = \sqrt{(1-e^2)}, \tag{7}$$

$$\tfrac{1}{9}\,\frac{d^2s_{,}}{d\theta^2} + \frac{s_{,}-1+e^2}{s_{,}^3} = 0; \tag{8}$$

i. e. a Keplerian ellipse with mean motion $3(n'-n)$, semimajor axis $a_{,}$, and eccentricity e; $a_{,}$ is a constant, and e a function of η only, satisfying the condition that the coefficient of $\cos 3\theta$ in δs is zero.

Neglecting powers of e greater than the fourth, we may write

$$\begin{aligned} s_{,} = 1 &+ \tfrac{1}{2} e^2 \\ &+ e(1 - \tfrac{3}{8} e^2) \cos 3\theta \\ &- \tfrac{1}{2} e^2 (1 - \tfrac{2}{3} e^2) \cos 6\theta \\ &+ \tfrac{3}{8} e^3 \cos 9\theta \\ &- \tfrac{1}{3} e^4 \cos 12\theta, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \frac{\sqrt{(1 - e^2)}}{s_{,}^2} = 1 &- 2e(1 - \tfrac{1}{8} e^2) \cos 3\theta \\ &+ \tfrac{5}{2} e^2 (1 - \tfrac{11}{30} e^2) \cos 6\theta \\ &- \tfrac{13}{4} e^3 \cos 9\theta \\ &+ \tfrac{103}{24} e^4 \cos 12\theta. \end{aligned}$$

Subtracting (8) from (6), neglecting powers and products of $\delta s = s - s_{,}$ and $\delta \nu = \nu - e^2$, and putting

$$\beta = \frac{3(1 - e^2)}{s_{,}^4} - \frac{2}{s_{,}^3},$$

we have

$$\begin{aligned} \mu \frac{d^2 \delta s}{d\theta^2} + \delta\mu \frac{d^2 s_{,}}{d\theta^2} + \beta \delta s + \frac{\delta \nu}{s_{,}^3} &= m' P \\ &= m' \sum_{0}^{\infty}{}_{i} (P_i \cos i\theta + Q_i \sin i\theta), \end{aligned} \tag{10}$$

in which $\delta\mu = \mu - \frac{1}{9}$, $\delta\nu$ satisfies the condition that δs shall contain no term independent of θ, i is any positive whole number, and P_i and Q_i are assumed to be functions of η only.

3. Considering, for the present, coefficients of cos 0θ, cos 3θ, cos 6θ, and cos 9θ only, we may write

$$\begin{aligned} \beta = 1 &+ 3e^2 \\ &- 6e(1 + \tfrac{9}{8} e^2) \cos 3\theta \\ &+ 12 e^2 \cos 6\theta \\ &- \tfrac{85}{4} e^3 \cos 9\theta. \end{aligned}$$

If we assume

$$\delta s = \sum_{1}^{\infty}{}_{i} (a_i \cos i\theta + g_i \sin i\theta), \tag{11}$$

and substitute for $s_,$ and δs their values given by (9) and (11), remembering that $a_3 = 0$, we have

$$
\begin{aligned}
\frac{d^2\delta s}{d\theta^2} &= -\,36a_6 \cos 6\theta - 81a_9 \cos 9\theta,\\
\frac{d^2 s_,}{d\theta^2} &= -\,9e(1 - \tfrac{3}{8}e^2)\cos 3\theta + 18e^2 \cos 6\theta,\\
\beta\delta s &= \quad 6e^2 a_6\\
&\quad - [3e(1 + \tfrac{14}{3}e^2)a_6 - 6e^2 a_9] \cos 3\theta\\
&\quad + [(1 + 3e^2)a_6 - 3ea_9] \cos 6\theta\\
&\quad - (3ea_6 - a_9) \cos 9\theta,\\
\frac{1}{s_,^3} &= \quad 1 + \tfrac{3}{2}e^2\\
&\quad - 3e(1 + \tfrac{9}{8}e^2)\cos 3\theta\\
&\quad + \tfrac{9}{2}e^2 \cos 6\theta.
\end{aligned}
$$

Substituting in (10) and separating by indeterminate coefficients, we obtain

$$
\begin{aligned}
6e^2 a_6 + (1 + \tfrac{3}{2}e^2)\delta\nu &= m'P_0,\\
-\,9e(1 - \tfrac{3}{8}e^2)\delta\mu - 3e(1 + \tfrac{14}{3}e^2)a_6 + 6e^2 a_9 - 3e(1 + \tfrac{9}{8}e^2)\delta\nu &= m'P_3,\\
18e^2\delta\mu - 3(1 - e^2 + 12\delta\mu)a_6 - 3ea_9 + \tfrac{9}{2}e^2\delta\nu &= m'P_6,\\
-\,3ea_6 - 8a_9 &= m'P_9;
\end{aligned}
$$

or

$$
\begin{aligned}
\delta\nu &= \quad (1 - \tfrac{3}{2}e^2)m'P_0 - 6e^2 a_6,\\
9e\delta\mu &= -\,3e(1 + \tfrac{3}{2}e^2)\delta\nu - (1 + \tfrac{3}{8}e^2)m'P_3 - 3e(1 + \tfrac{121}{24}e^2)a_6 + 6e^2 a_9,\\
a_6 &= \quad \tfrac{3}{2}e^2\delta\nu + 6e^2\delta\mu - \tfrac{1}{3}(1 + e^2 - 12\delta\mu)m'P_6 - ea_9,\\
a_9 &= -\,\tfrac{3}{8}ea_6 - \tfrac{1}{8}m'P_9;
\end{aligned}
$$

whence, solving by trial,

$$
\begin{aligned}
m'^{-1}\delta\nu &= \quad (1 - \tfrac{3}{2}e^2)P_0 + 2e^2 P_6,\\
9m'^{-1}\delta\mu . e &= -\,3e(1 - \tfrac{1}{2}e^2)P_0 - (1 - \tfrac{13}{8}e^2)P_3 + e(1 - \tfrac{5}{6}e^2 + 12\delta\mu)P_6 - \tfrac{9}{8}e^2 P_9,\\
m'^{-1}a_6 &= -\,\tfrac{1}{2}e^2 P_0 - \tfrac{2}{3}eP_3 - \tfrac{1}{3}(1 - \tfrac{5}{8}e^2 - 12\delta\mu)P_6 + \tfrac{1}{8}eP_9, \qquad (12)\\
m'^{-1}a_9 &= \quad \tfrac{1}{8}eP_6 - \tfrac{1}{8}P_9.
\end{aligned}
$$

For the succeeding terms, we may write

$$a_{3i} = \frac{1}{1-i^2} m' P_{3i}. \tag{13}$$

4. Considering cosine terms which involve only those multiples of θ which are not also multiples of 3θ, equation (10) may be written, with sufficient accuracy,

$$\tfrac{1}{9} \frac{d^2\delta s}{d\theta} + \beta\delta s = m' \Sigma_i P_i \cos i\theta,$$

in which we may assume

$$\delta s = a_1 \cos \theta + a_2 \cos 2\theta + a_4 \cos 4\theta + \dots$$

and

$$\beta = 1 + 2\beta_3 \cos 3\theta + 2\beta_6 \cos 6\theta;$$

whence, separating by indeterminate coefficients, we have

$$\begin{aligned}
&\tfrac{8}{9} a_1 + \beta_3 a_2 + \beta_3 a_4 + \beta_6 a_5 + \beta_6 a_7 = m' P_1, \\
&\beta_3 a_1 + \tfrac{5}{9} a_2 + \beta_6 a_4 + \beta_3 a_5 + \beta_6 a_8 = m' P_2, \\
&\beta_3 a_1 + \beta_6 a_2 - \tfrac{7}{9} a_4 + \beta_3 a_7 + \beta_6 a_{10} = m' P_4, \\
&\beta_6 a_1 + \beta_3 a_2 - \tfrac{16}{9} a_5 + \beta_3 a_8 + \beta_6 a_{11} = m' P_5, \\
&\beta_6 a_1 + \beta_3 a_4 - \tfrac{40}{9} a_7 + \beta_3 a_{10} + \beta_6 a_{13} = m' P_7, \\
&\beta_6 a_2 + \beta_3 a_5 - \tfrac{55}{9} a_8 + \beta_3 a_{11} + \beta_6 a_{14} = m' P_8, \\
&\dots\dots\dots\dots\dots\dots\dots
\end{aligned} \tag{14}$$

Putting $p_i = m' P_i$, and calling Δ the determinant whose constituents are the coefficients of a_1, a_2, etc.; Δ_1 the determinant formed by substituting p_1, p_2, etc. for the first column in Δ, i. e. for the coefficients of a_1; Δ_2 the determinant formed by substituting p_1, p_2, etc. for the coefficients of a_2, etc., we have

$$a_1 = \frac{\Delta_1}{\Delta}; \quad a_2 = \frac{\Delta_2}{\Delta}; \quad a_4 = \frac{\Delta_4}{\Delta}; \quad \text{etc.}$$

To expand these determinants, let us consider the determinant in its most general form. If we put

$$A_{ijk\dots} = \frac{1}{a_{ii}\, a_{jj}\, a_{kk} \dots} \left| \begin{array}{cccc} 0 & a_{ji} & a_{ki} & \dots \\ a_{ij} & 0 & a_{kj} & \dots \\ a_{ik} & a_{jk} & 0 & \dots \\ \dots & \dots & \dots & \dots \end{array} \right|,$$

in which $i < j < k < \ldots$, we have

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} & \ldots \\ a_{12} & a_{22} & a_{32} & \ldots \\ a_{13} & a_{23} & a_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = a_{11}\, a_{22}\, a_{33} \ldots (1 + \Sigma A_{ij} + \Sigma A_{ijk} + \Sigma A_{ijkl} + \ldots), \quad (15)$$

as may readily be seen by transforming and generalizing the method for expanding in products of the leading constituents given in Burnside and Panton's Theory of Equations, p. 246.

In the expansion of Δ we find

$$\Sigma A_{ij} = - A_2 \beta_3^{\,2} + \ldots,$$

in which

$$\begin{aligned} A_2 = {} & \tfrac{9}{8} \cdot \tfrac{9}{5} - \tfrac{9}{5} \cdot \tfrac{9}{16} + \tfrac{9}{16} \cdot \tfrac{9}{55} + \ldots \\ & - \tfrac{9}{8} \cdot \tfrac{9}{7} + \tfrac{9}{7} \cdot \tfrac{9}{40} + \tfrac{9}{40} \cdot \tfrac{9}{91} + \ldots \\ = {} & \sum_0^{\infty}{}'_i \left(\frac{1}{1 - (i - \frac{1}{3})^2} \cdot \frac{1}{1 - (i + \frac{2}{3})^2} + \frac{1}{1 - (i + \frac{1}{3})^2} \cdot \frac{1}{1 - (i + \frac{4}{3})^2} \right) \\ = {} & \tfrac{1}{6} \sum_0^{\infty}{}_i \left\{ \begin{array}{l} \dfrac{3}{1 + (i - \frac{1}{3})} - \dfrac{1}{1 - (i - \frac{1}{3})} - \dfrac{1}{1 + (i + \frac{2}{3})} + \dfrac{3}{1 - (i + \frac{2}{3})} \\ + \dfrac{3}{1 + (i + \frac{1}{3})} - \dfrac{1}{1 - (i + \frac{1}{3})} - \dfrac{1}{1 + (i + \frac{4}{3})} + \dfrac{3}{1 - (i + \frac{4}{3})} \end{array} \right\} \\ = {} & 0; \end{aligned}$$

whence, with sufficient accuracy,

$$\begin{aligned} \Delta & = \tfrac{8}{9} \cdot \tfrac{5}{9} \cdot - \tfrac{7}{9} \cdot - \tfrac{16}{9} \ldots \\ & = \tfrac{8}{9} \prod_1^{\infty}{}_i \overline{\left[1 - (i - \tfrac{1}{3})^2\right] \left[1 - (i + \tfrac{1}{3})^2\right]}. \end{aligned}$$

Expanding Δ_1, we obtain

$$\Delta_1 = p_1 \cdot \tfrac{5}{9} \cdot - \tfrac{7}{9} \cdot - \tfrac{16}{9} \ldots \left(\begin{array}{l} 1 + \tfrac{9}{8} \left(\tfrac{9}{5} - \tfrac{9}{7}\right) \beta_3^{\,2} \\ \quad - \tfrac{9}{5}\, \beta_3 \dfrac{p_2}{p_1} + \tfrac{9}{7}\, \beta_3 \dfrac{p_4}{p_1} \\ \quad + \tfrac{9}{16}\, \beta_6 \dfrac{p_5}{p_1} + \tfrac{9}{40}\, \beta_6 \dfrac{p_7}{p_1} \end{array} \right);$$

whence

$$\begin{aligned} m'^{-1}a_1 = {} & \tfrac{9}{8}\left[1 + \tfrac{9}{8}\left(\tfrac{9}{5} - \tfrac{9}{7}\right)\beta_3^2\right]P_1 \\ & - \tfrac{9}{8}\cdot\tfrac{9}{5}\,\beta_3 P_2 + \tfrac{9}{8}\cdot\tfrac{9}{7}\,\beta_3 P_4 \\ & + \tfrac{9}{8}\cdot\tfrac{9}{16}\,\beta_6 P_5 + \tfrac{9}{8}\cdot\tfrac{9}{40}\,\beta_6 P_7, \end{aligned}$$

in which the coefficient of β_3^2 is equal to A_2 after the omission of those terms in A_2 which contain as factors the reciprocals of such constituents as are situated at the intersection of the p-column and the diagonal; in other words, since $A_2 = 0$, it is equal to the sum of the terms mentioned taken with opposite signs.

In general, when i is not a multiple of 3, we have

$$\begin{aligned} m'^{-1}a_i = {} & \frac{9}{9-i^2}\left[1 + \frac{9}{9-i^2}\left\{\frac{9}{9-(i-3)^2} + \frac{9}{9-(i+3)^2}\right\}\beta_3^2\right]P_i \\ & - \frac{9}{9-i^2}\cdot\frac{9}{9-(i-3)^2}\,\beta_3 P_{i-3} - \frac{9}{9-i^2}\cdot\frac{9}{9-(i+3)^2}\,\beta_3 P_{i+3} \qquad (16) \\ & - \frac{9}{9-i^2}\cdot\frac{9}{9-(i-6)^2}\,\beta_6 P_{i-6} - \frac{9}{9-i^2}\cdot\frac{9}{9-(i+6)^2}\,\beta_6 P_{i+6}, \end{aligned}$$

in which

$$\begin{aligned} P_{-j} &= P_{+j}, \\ \beta_3 &= -3e(1 + \tfrac{9}{8}e^2), \\ \beta_6 &= 6e^2. \end{aligned}$$

5. For the terms involving only sines of multiples of 3θ, equation (10) may be written

$$\mu\frac{d^2\delta s}{d\theta^2} + \beta\delta s = m'\sum_1^\infty{}_i Q_{3i}\sin 3i\theta,$$

in which we may assume

$$\delta s = \sum_1^\infty{}_i\, g_{3i}\sin 3i\theta,$$

$$\beta = \beta_0 + 2\sum_1^\infty{}_i\beta_{3i}\cos 3i\theta,$$

$$m'Q_{3i} = q_{3i};$$

whence, separating by indeterminate coefficients, we have

$$\begin{aligned} \lambda_3 g_3 + (\beta_3 - \beta_9)g_6 + (\beta_6 - \beta_{12})g_9 + (\beta_9 - \beta_{15})g_{12} + \ldots &= q_3, \\ (\beta_3 - \beta_9)g_3 + \lambda_6 g_6 + (\beta_3 - \beta_{15})g_9 + (\beta_6 - \beta_{18})g_{12} + \ldots &= q_6, \end{aligned}$$

$$(\beta_6-\beta_{12})g_3+(\beta_3-\beta_{15})g_6+\qquad\lambda_9 g_9+(\beta_3-\beta_{21})g_{12}+\ldots=q_9,$$
$$(\beta_9-\beta_{15})g_3+(\beta_6-\beta_{18})g_6+(\beta_3-\beta_{21})g_9+\qquad\lambda_{12}g_{12}+\ldots=q_{12},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;;$$

where, if $\sigma=9\delta\mu$,

$$\begin{aligned}\lambda_3&=-3e^2-\sigma+\tfrac{11}{8}e^4,\\ \lambda_6&=-3+3e^2-4\sigma,\\ \lambda_9&=-8,\\ \lambda_{12}&=-15.\end{aligned}$$

If D be the determinant whose constituents are the coefficients of g_3, g_6, etc.; D_3 the determinant found by substituting q_3, q_6, etc. for the first column in D, i. e. for the coefficients of g_3, etc., we have, as in the preceding section,

$$g_3=\frac{D_3}{D};\qquad g_6=\frac{D_6}{D};\qquad \text{etc.}$$

Applying formula (15), we obtain

$$\begin{aligned}
D&=\lambda_3\lambda_6\lambda_9\ldots\left(1-\frac{(\beta_3-\beta_9)^2}{\lambda_3\lambda_6}-\frac{\beta_6{}^2}{\lambda_3\lambda_9}-\frac{\beta_3{}^2}{\lambda_6\lambda_9}-\frac{\beta_3{}^2}{\lambda_9\lambda_{12}}+\frac{2\beta_3{}^2\beta_6}{\lambda_3\lambda_6\lambda_9}+\frac{\beta_3{}^4}{\lambda_3\lambda_6\lambda_9\lambda_{12}}\right),\\
D_3&=q_3\lambda_6\lambda_9\ldots\left(1-\frac{\beta_3q_6}{\lambda_6q_3}-\frac{\beta_6q_9}{\lambda_9q_3}-\frac{\beta_3{}^2}{\lambda_6\lambda_9}-\frac{\beta_3{}^2}{\lambda_9\lambda_{12}}+\frac{2\beta_3{}^2q_9}{\lambda_6\lambda_9q_3}\right),\\
D_6&=\lambda_3q_6\lambda_9\ldots\left(1-\frac{\beta_3q_3}{\lambda_3q_6}\right),\\
D_9&=\lambda_3\lambda_6q_9\lambda_{12}\ldots\left(1-\frac{\beta_3{}^2}{\lambda_3\lambda_6}-\frac{\beta_6q_3}{\lambda_3q_9}+\frac{2\beta_3{}^2q_3}{\lambda_3\lambda_6q_9}\right),\\
D_{12}&=\lambda_3\lambda_6\lambda_9q_{12}\lambda_{15}\ldots\left(1-\frac{\beta_3{}^2}{\lambda_3\lambda_6}\right),\\
&\ldots\ldots\ldots\ldots\ldots\;;
\end{aligned}$$

whence, substituting for λ_3, β_3, etc. within the parentheses their values in terms of σ and e,

$$\begin{aligned}
D&=-\sigma\lambda_6\lambda_9\ldots(1+\tfrac{71}{20}e^2),\\
D_3&=\lambda_6\lambda_9\ldots[(1-\tfrac{9}{20}e^2)q_3-eq_6+\tfrac{3}{2}q_9],\\
D_6&=\lambda_9\lambda_{12}\ldots[3eq_3-(3e^2+\sigma)q_6],\\
D_9&=\lambda_6\lambda_{12}\lambda_{15}\ldots(-12e^2q_3-\sigma q_9),\\
D_{12}&=\lambda_6\lambda_9\lambda_{15}\lambda_{18}\ldots(-\sigma q_{12}),\\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Substituting $m'Q_{3i}$ for q_{3i} and dividing by $m'D/\sigma$, we have

$$\begin{aligned} m'^{-1}\sigma g_3 &= -(1-4e^2)Q_3 + eQ_6 - \tfrac{3}{2}e^2Q_9, \\ m'^{-1}\sigma g_6 &= \quad eQ_3 - (e^2 + \tfrac{1}{3}\sigma)Q_6, \\ m'^{-1}\sigma g_9 &= -\tfrac{3}{2}e^2Q_3 - \tfrac{1}{8}\sigma Q_9; \end{aligned} \tag{17}$$

and, for succeeding terms,

$$g_{3i} = \lambda_{3i}^{-1}Q_{3i} = \frac{1}{1-i^2}\,m'Q_{3i}. \tag{18}$$

6. For the sine terms not involving multiples of 3θ, we may assume $\mu = \frac{1}{9}$ and $\beta = 1$; whence, substituting in (10) and separating by intederminate coefficients,

$$g_i = \frac{9}{9-i^2}\,m'Q_i. \tag{19}$$

7. Instead of (3), we may write

$$\begin{aligned} s^2\frac{dw}{\mathrm{n}dt} &= \sqrt{(1-\nu)} + \frac{m'}{\sqrt{a}}\,\frac{n'}{n'-n}\,U \\ &= \sqrt{(1-\nu)} + m'\sum_{1}^{\infty}{}_i(S_i\cos i\theta + T_i\sin i\theta), \end{aligned} \tag{20}$$

in which S_i and T_i are assumed to be functions of η only. We have, also,

$$s_{,}^2\frac{dw_{,}}{3d\theta} = \sqrt{(1-e^2)}. \tag{7}$$

If we subtract (7) from (20), neglecting powers and products of δs, $\delta\nu = \nu - e^2$, and

$$\delta\frac{dw}{\mathrm{n}dt} = \frac{dw}{\mathrm{n}dt} - \frac{dw_{,}}{3d\theta},$$

we obtain

$$2s_{,}\frac{dw_{,}}{3d\theta}\,\delta s + s_{,}^2\delta\frac{dw}{\mathrm{n}dt} = -\tfrac{1}{2}(1+\tfrac{1}{2}e^2)\,\delta\nu + m'\sum_{1}^{\infty}{}_i(S_i\cos i\theta + T_i\sin i\theta), \tag{21}$$

in which

$$s_{,}\frac{dw_{,}}{3d\theta} = 1 - \tfrac{1}{2}e^2 - e\cos 3\theta + e^2\cos 6\theta,$$

$$s_{,}^2 = 1 + \tfrac{3}{2}e^2 + 2e\cos 3\theta - \tfrac{1}{2}e^2\cos 6\theta.$$

Assuming

$$\delta\frac{dw}{\mathrm{n}dt} = \sum_{0}^{\infty}{}_i(n_i\cos i\theta + h_i\sin i\theta), \tag{22}$$

in which n_i and h_i are functions of η only, substituting in (21), and separating by indeterminate coefficients, we have, approximately,

$$\begin{aligned}
e^2a_6 &+ (1 + \tfrac{3}{2}e^2)n_0 + en_3 - \tfrac{1}{4}e^2n_6 = -\tfrac{1}{2}(1 + \tfrac{1}{2}e^2)\delta\nu,\\
-ea_6 &+ 2en_0 + n_3 + en_6 = m'S_3,\\
2a_6 &+ n_6 = m'S_6;
\end{aligned}$$

the solution of which gives

$$\begin{aligned}
n_0 &= -\tfrac{1}{2}(1 + e^2)\delta\nu - \tfrac{9}{2}e^2a_6 - em'S_3 + \tfrac{5}{4}e^2m'S_6,\\
n_3 &= e\delta\nu + 3ea_6 + m'S_3 - em'S_6,\\
n_6 &= -2a_6 + m'S_6;
\end{aligned}\tag{23}$$

or, substituting from equations (12),

$$\begin{aligned}
m'^{-1}n_0 &= -\tfrac{1}{2}(1 - \tfrac{1}{2}e^2)P_0 + \tfrac{1}{2}e^2P_6 - eS_3 + \tfrac{5}{4}e^2S_6,\\
m'^{-1}n_3 &= eP_0 - eP_6 + S_3 - eS_6,\\
m'^{-1}n_6 &= \tfrac{2}{3}P_6 + S_6.
\end{aligned}\tag{23$'$}$$

Also,

$$\begin{aligned}
2(1 - e^2)g_3 - eg_6 &+ (1 + \tfrac{7}{4}e^2)h_3 + eh_6 = m'T_3,\\
-eg_3 + 2g_6 &+ eh_3 + h_6 = m'T_6,\\
2g_9 &- \tfrac{1}{4}e^2h_3 + eh_6 + h_9 = m'T_9;
\end{aligned}\tag{24}$$

whence

$$\begin{aligned}
h_3 &= m'T_3 - 2(1 - \tfrac{5}{4}e^2)g_3 + 3eg_6,\\
h_6 &= m'T_6 + 3eg_3 - 2g_6,\\
h_9 &= m'T_9 - \tfrac{7}{2}e^2g_3 + 2eg_6 - 2g_9;
\end{aligned}\tag{24$'$}$$

or, substituting from equations (17),

$$\begin{aligned}
m'^{-1}\sigma h_3 &= \sigma T_3 + 2(1 - \tfrac{15}{4}e^2)Q_3 - 2eQ_6 + 3e^2Q_9,\\
m'^{-1}\sigma h_6 &= \sigma T_6 - 5eQ_3 + 3e^2Q_6,\\
m'^{-1}\sigma h_9 &= \sigma T_9 + \tfrac{17}{2}e^2Q_3.
\end{aligned}$$

For the remaining terms, we may write, in general,

$$2a_i - ea_{i+3} - ea_{i-3} + n_i + en_{i+3} + en_{i-3} = m'S_i, \tag{25}$$

$$2g_i - eg_{i+3} - eg_{i-3} + h_i + eh_{i+3} + eh_{i-3} = m'T_i, \tag{26}$$

in which

$$a_{i-3} = a_{-i+3}, \qquad n_{i-3} = n_{-i+3},$$

$$g_{i-3} = -g_{-i+3}, \qquad h_{i-3} = -h_{-i+3};$$

whence, approximately,

$$n_{i\pm 3} = m'S_{i\pm 3} - 2a_{i\pm 3},$$

$$h_{i\pm 3} = m'T_{i\pm 3} - 2g_{i\pm 3};$$

which, substituted in (25) and (26), give

$$n_i = m'S_i - em'S_{i+3} - em'S_{i-3} - 2a_i + 3ea_{i+3} + 3ea_{i-3}, \tag{27}$$

$$h_i = m'T_i - em'T_{i+3} - em'T_{i-3} - 2g_i + 3eg_{i+3} + 3eg_{i-3}, \tag{28}$$

in which

$$S_{i-3} = S_{-i+3}, \qquad T_{i-3} = -T_{-i+3}.$$

8. If we omit periodic terms, we may write

$$\left(\frac{dw}{\mathrm{n}dt}\right)_0 = \left(\frac{dw_{,}}{3d\theta}\right)_0 + \left(\delta\frac{dw}{\mathrm{n}dt}\right)_0,$$

or

$$\frac{n}{\mathrm{n}} = 1 + Am', \tag{29}$$

in which A is the constant part of $m'^{-1}n_0$, which is given by the first of equations (23′). We may also put

$$\sigma = 9\delta\mu = 9\,\frac{(n' - n)^2}{\mathrm{n}^2} - 1 = Bm', \tag{30}$$

in which B is obtained from the second of equations (12).

Putting

$$\gamma = \frac{3(n' - n)}{n} - 1 = \frac{3n' - 4n}{n}, \tag{31}$$

and substituting in (30), we have

$$(1 + \gamma)^2 (1 + Am')^2 = 1 + Bm';$$

or, with sufficient accuracy,

$$2\gamma + \gamma^2 + 2Am' = Bm';$$

whence

$$m' = \frac{2\gamma + \gamma^2}{B - 2A}. \tag{32}$$

9. We may now put

$$\frac{dw}{\mathrm{n}dt} = \frac{n}{\mathrm{n}} + n_\eta + \left(\frac{dw}{\mathrm{n}dt}\right)_\theta$$

$$= \frac{n}{\mathrm{n}} + n_\eta + \left(\frac{dw_,}{3d\theta}\right)_\theta + \left(\delta\frac{dw}{\mathrm{n}dt}\right)_\theta, \qquad (33)$$

in which

$$n_\eta = n_0 - Am'.$$

The subscript θ indicates that only terms involving that quantity are considered.

Multiplying by $\mathrm{n}dt$ and integrating,

$$w = w_0 + nt + \frac{\mathrm{n}}{4n - 3n'}\int n_\eta d\eta + \frac{\mathrm{n}}{3(n'-n)}(w_,)_\theta + \frac{\mathrm{n}}{n'-n}\int\left(\delta\frac{dw}{\mathrm{n}dt}\right)_\theta d\theta; \quad (34)$$

or, with sufficient accuracy,

$$w = w_0 + nt - \frac{1}{\gamma}\int n_\eta d\eta + (1 - Am' - \gamma)(w_,)_\theta + 3\int\left(\delta\frac{dw}{\mathrm{n}dt}\right)_\theta d\theta. \qquad (34')$$

Since

$$d\eta = \mathrm{g}d\theta,$$

where

$$\mathrm{g} = \frac{4n - 3n'}{n' - n} = -\frac{3\gamma}{1 + \gamma},$$

we have, by integration,

$$\eta = \mathrm{g}\theta + \mathrm{g}_0;$$

whence

$$\cos j\eta \cos i\theta = \tfrac{1}{2}\cos(i\theta + j\eta) + \tfrac{1}{2}\cos(i\theta - j\eta)$$

$$= \tfrac{1}{2}\cos[(i + j\mathrm{g})\theta + j\mathrm{g}_0] + \tfrac{1}{2}\cos[(i - j\mathrm{g})\theta - j\mathrm{g}_0],$$

and

$$\int\cos j\eta \cos i\theta\, d\theta = \tfrac{1}{2}\frac{1}{i + j\mathrm{g}}\sin(i\theta + j\eta) + \tfrac{1}{2}\frac{1}{i - j\mathrm{g}}\sin(i\theta - j\eta)$$

$$= \frac{i}{i^2 - j^2\mathrm{g}}\cos j\eta \sin i\theta - \frac{j\mathrm{g}}{i^2 - j^2\mathrm{g}^2}\sin j\eta \cos i\theta. \qquad (35)$$

Similarly

$$\int\sin j\eta \cos i\theta\, d\theta = \frac{i}{i^2 - j^2\mathrm{g}^2}\sin j\eta \sin i\theta + \frac{j\mathrm{g}}{i^2 - j^2\mathrm{g}^2}\cos j\eta \cos i\theta, \quad (36)$$

$$\int\cos j\eta \sin i\theta\, d\theta = -\frac{i}{i^2 - j^2\mathrm{g}^2}\cos j\eta \cos i\theta - \frac{j\mathrm{g}}{i^2 - j^2\mathrm{g}^2}\sin j\eta \sin i\theta, \quad (37)$$

$$\int\sin j\eta \sin i\theta\, d\theta = -\frac{i}{i^2 - j^2\mathrm{g}^2}\sin j\eta \cos i\theta + \frac{j\mathrm{g}}{i^2 - j^2\mathrm{g}^2}\cos j\eta \sin i\theta. \quad (38)$$

Since, however, g is small and is multiplied by small coefficients, it may, in general, be neglected; whence (34′) becomes

$$w = w_0 + nt - \frac{1}{\gamma}\int n_\eta d\eta + (1 - Am' - \gamma)(w_{,})_\theta + 3\sum_{1}^{\infty}{}_i \frac{1}{i}(n_i \sin i\theta - h_i \cos i\theta). \qquad (34'')$$

Similarly, in integrating $Srd\theta$, the coefficients of sines and cosines of multiples of θ may, in general, be considered as constant.

10. Differentiating

$$\sqrt{(1-\nu)} = \sqrt{(1-\nu_0)} + \frac{m'}{\sqrt{a}}\frac{n'}{4n-3n'}\int (Sr)_0 d\eta \qquad (39)$$

with regard to η, we have

$$-\tfrac{1}{2}(1+\tfrac{1}{2}\nu)\frac{d\nu}{d\eta} = -e(1+\tfrac{1}{2}e^2)\frac{de}{d\eta} - \tfrac{1}{2}\frac{d\delta\nu}{d\eta} = \frac{m'}{\sqrt{a}}\frac{n'}{4n-3n'}(Sr)_0,$$

or

$$(Sr)_0 = -\frac{\sqrt{a}}{m'}\frac{4n-3n'}{n'}\left[e(1+\tfrac{1}{2}e^2)\frac{de}{d\eta} + \tfrac{1}{2}\frac{d\delta\nu}{d\eta}\right], \qquad (40)$$

and may assume

$$e = e_0(1 + e_1\cos\eta + f_1\sin\eta + e_2\cos 2\eta + f_2\sin 2\eta + \ldots), \qquad (41)$$

in which e_0, e_1, f_1, etc. are constants.

11. The coefficients of sines and cosines of multiples of η in e, $\delta\nu$, a_i, n_i, g_i, and h_i are found by substituting (41) in equations (12), (17), and (23)–(26), and separating by indeterminate coefficients. With assumed values of e_0, P_0, P_3, etc., the numerical values of B and of the coefficients in the expression for e/e_0 are first obtained from the second of equations (12); the numerical values of the other coefficients are then found by substituting these in the remainder of the equations mentioned.

12. For the computation of the forces we have the well-known formulæ,

$$\rho^2 = r'^2\sin^2(w'-w) + [r'\cos(w'-w) - r]^2,$$

$$h = \frac{1}{\rho^3} - \frac{1}{r'^3},$$

$$R = hr'\cos(w'-w) - \frac{r}{\rho^3},$$

$$S = hr'\sin(w'-w),$$

in which ρ is the distance between the two satellites, and r' and w' are the radius vector and longitude in orbit of Titan.

Assuming an approximate orbit, a^2R and Sr are computed for different values of η and θ. In so doing η and θ may be treated as constants. The assumption, however, that η is constant is equivalent to saying that the conjunction point is stationary, or, in other words, that the mean motions of the two satellites are commensurable. If t_c be the mean time of conjunction of the two satellites, we have

$$\theta = l' - l - \lambda = \mathrm{l}' - \mathrm{l} - \lambda,$$

in which

$$\mathrm{l} = \eta + 3(n' - n)\,(t - t_c)$$

and

$$\mathrm{l}' = \eta + 4(n' - n)\,(t - t_c)$$

are the mean longitudes of Hyperion and Titan on the assumption that η is constant.

Having determined the values of a^2R and Sr for individual values of θ and η, the former are expressed in series of sines and cosines of the latter by means of mechanical quadratures. With the values of a^2R and Sr thus expressed P_i, Q_i, S_i, T_i, are obtained and substituted in equations (12), (17), and (23)-(26), whence are derived the inequalities in r and w. The problem is solved when we have found expressions for r and w which agree with the assumed values of those quantities. The value of $(Sr)_0$ found by mechanical quadratures must also agree with that obtained by means of equation (40).

13. Assuming the perisaturnium of Titan as the origin of longitudes; putting

$$\lambda = 0,$$

$$n = 16^\circ.9199, \qquad n' = 22^\circ.5770,$$

$$\nu = e^2 = 0.01, \qquad e' = 0.0284,$$

$$\log a = 0.08351, \qquad a' = 1;$$

and neglecting δs and $\delta \frac{dw}{\mathrm{n}dt}$, the following values of a^2R and Sr were obtained for $\eta = 0$ and $\eta = 180^\circ$:—

θ	$\eta = 0^\circ$		$\eta = 180^\circ$	
	a^2R	Sr	a^2R	Sr
0°.0	− 12.64	0.00	− 17.20	0.00
7.5	− 8.17	+ 3.23	− 11.22	+ 4.71
15.0	− 4.32	+ 2.02	− 5.76	+ 3.42
22.5	− 2.83	+ 1.07	− 3.61	+ 1.87
30.0	− 2.15	+ 0.52	− 2.68	+ 0.91
37.5	− 1.81	+ 0.24	− 2.17	+ 0.35
45.0	− 1.63	+ 0.13	− 1.84	+ 0.02
52.5	− 1.56	+ 0.07	− 1.63	− 0.12
60.0	− 1.58	+ 0.01	− 1.42	− 0.25
67.5	− 1.52	− 0.11	− 1.28	− 0.33
75.0	− 1.40	− 0.33	− 1.14	− 0.44
82.5	− 1.18	− 0.58	− 0.98	− 0.54
90.0	− 0.86	− 0.85	− 0.78	− 0.72
97.5	− 0.46	− 1.09	− 0.57	− 0.81
105.0	− 0.12	− 1.09	− 0.28	− 1.00
112.5	+ 0.20	− 1.06	+ 0.03	− 1.08
120.0	+ 0.46	− 0.96	+ 0.34	− 1.06
127.5	+ 0.66	− 0.84	+ 0.67	− 0.96
135.0	+ 0.81	− 0.68	+ 0.93	− 0.77
142.5	+ 0.93	− 0.53	+ 1.09	− 0.53
150.0	+ 1.03	− 0.39	+ 1.15	− 0.31
157.5	+ 1.11	− 0.27	+ 1.14	− 0.15
165.0	+ 1.16	− 0.17	+ 1.11	− 0.06
172.5	+ 1.19	− 0.08	+ 1.07	− 0.02
180.0	+ 1.06	0.00	+ 1.21	0.00

For negative values of θ, the signs of Sr change, but not those of a^2R.

Considering terms involving multiples of 3θ only, the following values of a^2R and Sr give

$$\begin{aligned} a^2R = & -1.326 - 1.224\cos 3\theta - 0.847\cos 6\theta - 0.597\cos 9\theta - 0.323\cos 12\theta \\ & +(0.167 + 0.276\cos 3\theta + 0.158\cos 6\theta + 0.084\cos 9\theta + 0.043\cos 12\theta)\cos\eta \\ Sr = & \qquad + 0.633\sin 3\theta + 0.534\sin 6\theta + 0.356\sin 9\theta + 0.230\sin 12\theta \\ & \qquad +(0.103\sin 3\theta - 0.091\sin 6\theta - 0.069\sin 9\theta - 0.039\sin 12\theta)\cos\eta . \end{aligned}$$

We have, also,

$$\frac{\sqrt{(1-\nu)}}{3\sqrt{a}}\,\frac{n'}{n'-n} = 1.202$$

and

$$\frac{1}{s^3} = \frac{1}{s_{\prime}^{3}} = 1.015 - 0.303\cos 3\theta + 0.045\cos 6\theta - 0.007\cos 9\theta + 0.001\cos 12\theta ;$$

whence

$$P_0 = -1.108 + 0.132\cos\eta ,$$

$$P_3 = -2\ 709 + 0.516\cos\eta , \qquad S_3 = -0.765 + 0.124\cos\eta ,$$

$$P_6 = -1.225 + 0.221\cos\eta , \qquad S_6 = -0.322 + 0.054\cos\eta ,$$

$$P_9 = -0.806 + 0.130\cos\eta , \qquad S_9 = -0.145 + 0.029\cos\eta .$$

Substituting in the first of equations (23′), we have

$$m'^{-1}n_0 = 0.639 - 0.069\cos\eta ;$$

whence

$$A = 0.639 \qquad \text{and} \qquad m'^{-1}n_\eta = -0.069\cos\eta .$$

The assumed values of n and n' give $12\delta\mu = 0.0081$; whence, by the second of equations (12),

$$9m'^{-1}\delta\mu\,.\,e = 2.883 - 0.526\cos\eta ;$$

whence, assuming $e = \frac{1}{10}(1 + e_1\cos\eta)$, we have

$$B = 9m'^{-1}\delta\mu = 28.83$$

and

$$e_1 = -0.183 ;$$

i. e.

$$e = \quad 0.1000 - 0.0183\cos\eta .$$

We have, also,

$$3n' - 4n = 0^\circ.0514 ,$$

$$\gamma = 0.003038 ;$$

whence, substituting in (32),

$$m' = \frac{1}{4528},$$

and the long period inequality in (34″) becomes

$$-\frac{1}{\gamma}\int n_\eta\, d\eta = \frac{0.069 \times 57^\circ.3 \sin\eta}{0.003038 \times 4528} = 0^\circ.287 \sin\eta.$$

The terms in θ, 2θ, 4θ, and 5θ give

$$\begin{aligned} a^2R &= -\ 3.23\cos\theta - 1.31\cos 2\theta - 1.31\cos 4\theta - 1.98\cos 5\theta \\ &\quad + (0.38\cos\theta + 0.31\cos 2\theta + 0.21\cos 4\theta + 0.07\cos 5\theta)\cos\eta, \\ Sr &= -\ 0.44\sin\theta + 0.76\sin 2\theta + 0.39\sin 4\theta + 0.57\sin 5\theta \\ &\quad + (-\ 0.06\sin\theta - 0.06\sin 2\theta - 0.14\sin 4\theta - 0.17\sin 5\theta)\cos\eta; \end{aligned}$$

whence

$$\begin{aligned} P_1 &= +\ 0.46 + 0.39\cos\eta, & S_1 &= +\ 1.59 + 0.22\cos\eta, \\ P_2 &= -\ 4.46 + 0.13\cos\eta, & S_2 &= -\ 1.38 + 0.11\cos\eta, \\ P_4 &= -\ 2.47 + 0.15\cos\eta, & S_4 &= -\ 0.35 + 0.12\cos\eta, \\ P_5 &= -\ 2.29 + 0.19\cos\eta; & S_5 &= -\ 0.41 + 0.12\cos\eta. \end{aligned}$$

The resulting values of a_i and n_i are

$$\begin{aligned} a_1 &= -\ 0.0002 + 0.0001\cos\eta, & n_1 &= +\ 0.0004 - 0.0002\cos\eta, \\ a_2 &= -\ 0.0018 + 0.0001\cos\eta, & n_2 &= +\ 0.0034 - 0.0001\cos\eta, \\ a_3 &= \quad 0.0000, & n_3 &= -\ 0.0002, \\ a_4 &= +\ 0.0005 - 0.0001\cos\eta, & n_4 &= -\ 0.0012 + 0.0002\cos\eta, \\ a_5 &= +\ 0.0006, & n_5 &= -\ 0.0018 + 0.0001\cos\eta, \\ a_6 &= +\ 0.0001, & n_6 &= -\ 0.0003; \\ a_7 &= -\ 0.0001; \end{aligned}$$

whence, neglecting terms involving g_i and those containing both η and θ,

$$\begin{aligned} r = a(s_, + \delta s) = as_, &- 0.0001\cos\theta - 0.0022\cos 2\theta \\ &+ 0.0006\cos 4\theta + 0.0007\cos 5\theta + 0.0001\cos 6\theta \\ &- 0.0001\cos 7\theta. \end{aligned}$$

We have, also,

$$w_{,} = w_0 + nt - (2e - \tfrac{1}{4}e^3)\sin 3\theta + \tfrac{5}{4}e^2 \sin 6\theta - \tfrac{13}{12}e^3 \sin 9\theta + \tfrac{103}{96}e^4 \sin 12\theta,$$

in which 3θ is the mean anomaly counted from *apo*saturnium, and

$$(Am' + \gamma)\,(w_{,})_\theta = -\,2e\,(Am' + \gamma) \sin 3\theta$$

$$= (-\,0.0006 + 0.0001 \cos \eta) \sin 3\theta;$$

whence, neglecting terms involving h_i and those containing both η and θ, equation (34″) gives

$$w = w_{,} + 0^\circ.29 \sin \eta + 0^\circ.17 \sin \theta + 0^\circ.29 \sin 2\theta - 0^\circ.01 \sin 3\theta$$

$$- 0^\circ.05 \sin 4\theta - 0^\circ.06 \sin 5\theta - 0^\circ.01 \sin 6\theta.$$

14. For $\eta = 90^\circ$, the forces were computed on the same hypothesis as before, and also on the assumption that w contains the additional term,

$$\varDelta w = -\,h_3 \cos 3\theta = -\,2^\circ.50 \cos 3\theta.$$

The corresponding term in r is, approximately,

$$\varDelta r = ag_3 \sin 3\theta = -\,\tfrac{1}{2}ah_3 \sin 3\theta = -\,0.0264 \sin 3\theta.$$

For $\eta = 270^\circ$, when $\lambda = 0$, the numerical values of the forces are the same as for $\eta = 90^\circ$; except that the sine terms in a^2R and the cosine terms in Sr change sign.

The values of a^2R and Sr obtained for $\eta = 90^\circ$ are as follows:—

θ	$\Delta w = 0$		$\Delta w = -2^\circ.5 \cos 3\theta$	
	a^2R	Sr	a^2R	Sr
0	− 14.08	+ 1.87	− 12.81	+ 2.93
2.5	− 12.31	+ 3.47	− 10.77	+ 3.94
5	− 10.03	+ 4.11	− 8.69	+ 4.14
7.5	− 8.03	+ 4.05	− 6.98	+ 3.86
10	− 6.44	+ 3.65	− 5.69	+ 3.41
15	− 4.39	+ 2.67	− 4.07	+ 2.50
20	− 3.34	+ 1.86	− 3.19	+ 1.80
25	− 2.75	+ 1.29	− 2.69	+ 1.31
30	− 2.38	+ 0.89	− 2.38	+ 0.97
35	− 2.15	+ 0.62	− 2.17	+ 0.72
40	− 1.98	+ 0.41	− 2.04	+ 0.54
45	− 1.88	+ 0.25	− 1.95	+ 0.39
50	− 1.79	+ 0.13	− 1.87	+ 0.26
55	− 1.72	+ 0.02	− 1.81	+ 0.14
60	− 1.63	− 0.10	− 1.73	− 0.01
70	− 1.41	− 0.35	− 1.46	− 0.30
80	− 1.09	− 0.61	− 1.13	− 0.62
90	− 0.71	− 0.83	− 0.70	− 0.86
100	− 0.33	− 0.96	− 0.29	− 1.01
110	+ 0.02	− 1.01	+ 0.08	− 1.04
120	+ 0.35	− 0.98	+ 0.41	− 0.95
130	+ 0.67	− 0.87	+ 0.72	− 0.82
140	+ 0.94	− 0.65	+ 0.96	− 0.62
150	+ 1.14	− 0.40	+ 1.13	− 0.40
160	+ 1.20	− 0.19	+ 1.20	− 0.19
170	+ 1.19	− 0.02	+ 1.18	− 0.06

θ	$\varDelta w = 0$		$\varDelta w = -2^\circ.5 \cos 3\theta$	
	a^2R	Sr	a^2R	Sr
180	+ 1.13	+ 0.05	+ 1.13	+ 0.07
190	+ 1.09	+ 0.10	+ 1.09	+ 0.01
200	+ 1.07	+ 0.18	+ 1.07	+ 0.15
210	+ 1.04	+ 0.31	+ 1.05	+ 0.32
220	+ 0.98	+ 0.55	+ 0.97	+ 0.56
230	+ 0.82	+ 0.81	+ 0.78	+ 0.85
240	+ 0.47	+ 1.06	+ 0.42	+ 1.08
250	0.00	+ 1.15	− 0.06	+ 1.13
260	− 0.50	+ 1.01	− 0.55	+ 0.97
270	− 0.90	+ 0.73	− 0.93	+ 0.69
280	− 1.19	+ 0.41	− 1.15	+ 0.41
290	− 1.30	+ 0.21	− 1.23	+ 0.26
300	− 1.36	+ 0.13	− 1.28	+ 0.23
305	− 1.40	+ 0.12	− 1.32	+ 0.22
310	− 1.48	+ 0.11	− 1.40	+ 0.21
315	− 1.60	+ 0.07	− 1.53	+ 0.17
320	− 1.77	− 0.02	− 1.71	+ 0.08
325	− 2.04	− 0.20	− 1.99	− 0.11
330	− 2.40	− 0.50	− 2.39	− 0.43
335	− 2.94	− 0.98	− 2.98	− 0.94
340	− 3.83	− 1.70	− 3.98	− 1.67
345	− 5.49	− 2.70	− 6.02	− 2.71
350	− 8.65	− 3.52	− 9.44	− 3.25
352.5	− 10.89	− 3.35	− 11.71	− 2.66
355	− 13.12	− 2.31	− 13.54	− 1.12
357.5	− 14.43	− 0.35	− 13.98	+ 1.04

This table was completed by interpolation so as to contain values of the forces for every two and a half degrees of θ; whence, considering terms involving multiples of 3θ only, we have for $\Delta w = 0$, $\Delta r = 0$,

$$\begin{aligned} a^2R = & -1.287 - 1.168\cos 3\theta - 0.801\cos 6\theta - 0.489\cos 9\theta - 0.294\cos 12\theta \\ & +(0.059\sin 3\theta + 0.134\sin 6\theta + 0.131\sin 9\theta + 0.109\sin 12\theta)\sin\eta, \\ Sr = & \ 0.654\sin 3\theta + 0.506\sin 6\theta + 0.408\sin 9\theta + 0.214\sin 12\theta \\ & + 0.127\sin 15\theta + 0.072\sin 18\theta \\ & +(0.077 + 0.091\cos 3\theta + 0.089\cos 6\theta + 0.094\cos 9\theta + 0.078\cos 12\theta \\ & + 0.064\cos 15\theta + 0.047\cos 18\theta)\sin\eta; \end{aligned}$$

whence

$$\begin{aligned} Q_3 &= 0.261\sin\eta, \\ Q_6 &= 0.200\sin\eta, \\ Q_9 &= 0.186\sin\eta, \\ Q_{12} &= 0.141\sin\eta. \end{aligned}$$

T_3, T_6, etc., are negligible.

In the same manner, for $\Delta w = -2^\circ.5\cos 3\theta$, $\Delta r = -0.0264\sin 3\theta$,

$$\begin{aligned} a^2R = & -1.259 - 1.113\cos 3\theta - 0.743\cos 6\theta - 0.429\cos 9\theta - 0.250\cos 12\theta \\ & +(0.110\sin 3\theta + 0.227\sin 6\theta + 0.224\sin 9\theta + 0.180\sin 12\theta)\sin\eta, \\ Sr = & \ 0.635\sin 3\theta + 0.468\sin 6\theta + 0.372\sin 9\theta + 0.176\sin 12\theta \\ & + 0.093\sin 15\theta + 0.045\sin 18\theta \\ & +(0.128 + 0.126\cos 3\theta + 0.158\cos 6\theta + 0.159\cos 9\theta + 0.133\cos 12\theta \\ & + 0.100\cos 15\theta + 0.074\cos 18\theta)\sin\eta, \\ \frac{\sqrt{(1-\nu)}}{\sqrt{a}}\,\frac{n'}{n'-n}\,U = & -0.763\cos 3\theta - 0.281\cos 6\theta - 0.126\cos 9\theta - 0.053\cos 12\theta \\ & - 0.023\cos 15\theta - 0.009\cos 18\theta \\ & +(0.151\sin 3\theta + 0.095\sin 6\theta + 0.064\sin 9\theta + 0.040\sin 12\theta \\ & + 0.024\sin 15\theta + 0.015\sin 18\theta)\sin\eta. \end{aligned}$$

Multiplying the latter by

$$\begin{aligned} 2s^{-3} = & \ 2.030 - 0.606\cos 3\theta + 0.090\cos 6\theta - 0.014\cos 9\theta + 0.002\cos 12\theta \\ & +(0.158\sin 3\theta - 0.008\sin 6\theta)\sin\eta, \end{aligned}$$

we have

$$U' = (0.301 \sin 3\theta + 0.077 \sin 6\theta + 0.085 \sin 9\theta + 0.052 \sin 12\theta) \sin \eta;$$

and, adding a^2R,

$$Q_3 = 0.411 \sin \eta,$$

$$Q_6 = 0.304 \sin \eta,$$

$$Q_9 = 0.309 \sin \eta,$$

$$Q_{12} = 0.232 \sin \eta.$$

In the present case, in a first approximation, the values of Q_3, Q_6, etc., may be considered as depending on the value of Δw for $\theta = 0$.

The above values give, by means of (24′) and (34″), for $\eta = 90^\circ$,

Assumed.	Computed.
$\Delta w = -2^\circ.5 \cos 3\theta$,	$\Delta w = +0^\circ.287 - 1^\circ.471 \cos 3\theta + 0^\circ.195 \cos 6\theta$ $- 0^\circ.024 \cos 9\theta + 0^\circ.002 \cos 12\theta$, $= -1^\circ.011$, for $\theta = 0$,
$\Delta w = 0$,	$\Delta w = +0^\circ.287 - 0^\circ.929 \cos 3\theta + 0^\circ.123 \cos 6\theta$ $- 0^\circ.014 \cos 9\theta + 0^\circ.002 \cos 12\theta$, $= -0^\circ.531$, for $\theta = 0$;

whence, by interpolation, Δw (assumed) $= \Delta w$(computed) $= -0^\circ.657$, for $\theta = 0$.

Hence, approximately,

$$Q_3 = 0.300 \sin \eta,$$

$$Q_6 = 0.227 \sin \eta,$$

$$Q_9 = 0.218 \sin \eta,$$

$$Q_{12} = 0.165 \sin \eta,$$

$$\Delta w = 0^\circ.29 - 1^\circ.07 \cos 3\theta + 0^\circ.14 \cos 6\theta - 0^\circ.02 \cos 9\theta,$$

$$\Delta r = \quad - 0.0100 \sin 3\theta + 0.0010 \sin 6\theta - 0.0001 \sin 9\theta.$$

For $\eta = 270^\circ$ and $\lambda = 0$, the values of these inequalities are the same as those for $\eta = 90^\circ$, but with opposite signs.

In a similar manner, a^2R, Sr, Δw, and Δr may be found for other values of η, and the coefficients of the sines and cosines of multiples of θ expressed as periodic functions of that quantity (η) by means of mechanical quadratures.

15. With the values of the co-ordinates thus found more accurate values may be found by a repetition of the process described in the preceding pages; but the solution cannot be considered as even approaching completion until the value of λ is obtained by the comparison of an assumed orbit with observations made near conjunction, and distributed, if possible, over the whole of the nineteen years during which η makes a cycle of 360°.

It may be noted that $(Sr)_0$ as given by equation (40) is nearly proportional to $de^2/d\eta$, and, therefore, is liable to converge slowly, since e^2 converges more slowly than e, and $de^2/d\eta$ converges more slowly than e^2. Hence, the values of $(Sr)_0$ obtained for 90° and 270° are not sufficient to give, approximately even, the coefficient of sin η in that quantity. On the other hand, $-\frac{1}{\gamma}\int n_\eta d\eta$ converges more rapidly than n_η; whence it is probable that the value of that inequality may be derived with considerable accuracy from the forces obtained for $\eta = 0$ and $\eta = 180°$; but, in computing those forces the inequality in e should not be neglected, except in a first approximation.

16. The method here suggested may be readily extended to cases in which the disturbed and disturbing bodies do not move in coplanar orbits. The asteroids furnish quite a number of interesting cases of mean motions nearly commensurate with that of Jupiter. As examples may be mentioned (153) Hilda and (190) Ismene, whose mean motions are each approximately three halves that of Jupiter.